Problem Set 2 due September 16, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: (a) Compute the inverses of the matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix}$$

for various numbers a, b, c, a', b', c'. **You must use the Gauss-Jordan elimination procedure** outlined on page 11 of the Lecture notes (or page 86 of the textbook), that is, by starting from the augmented matrices $\begin{bmatrix} L & I \end{bmatrix}$ and $\begin{bmatrix} U & I \end{bmatrix}$. *(10 points)*

(b) Compute the inverse of the matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

for any three non-zero numbers d_1, d_2, d_3. *(5 points)*

(c) Consider the 3×3 matrix $A = LDU$, where L, D and U are as above. Write A^{-1} as a product of three matrices, and as a single matrix. Note that the formula for A^{-1} as a single matrix will not be particularly pretty, but it will give you some practice with matrix multiplication. *(10 points)*

Problem 2: (a) If:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad X' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \tag{1}$$

and $XX' = I$, then what is the formula for a', b', c', d' in terms of a, b, c, d? You can look this up. *(5 points)*

(b) Now prove the aforementioned formula, by directly showing that the products of the two matrices (XX' and $X'X$) in (1) are both the unit matrix I (when computing the product, you must replace a', b', c', d' by their aforementioned formulas in terms of a, b, c, d). *(5 points)*

(c) Now consider the following block matrix:

$$X = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

where A and B are invertible, square matrices, C is a (not necessarily square) matrix, and 0 is a (also not necessarily square) zero matrix. Find (and prove) a formula for X^{-1} in terms of A, B, C and their inverses. What does your formula say in the explicit case:

$$X = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 5 \end{bmatrix}$$

(make sure you say what A, B, C are in this case)? *(10 points)*

Problem 3: (a) Instead of doing row operations, one can do column operations on a matrix. For example, start with:

$$A = \begin{bmatrix} \boxed{1} & 2 & 0 \\ -1 & \boxed{\boxed{-1}} & 3 \\ 2 & -3 & -5 \end{bmatrix}$$

and add appropriate multiples of the first <u>column</u> to the second and third columns, in such a way that all entries to the right of the box vanish. Then add an appropriate multiple of the second column to the third column, so that all entries to the right of the double box vanish. Carry out this process by showing all the steps. *(10 points)*

(b) Describe each of the steps in the process above as multiplying A on the right with an appropriate matrix. *(10 points)*

Problem 4: (a) Consider the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

and write it as the sum of a symmetric matrix and an anti-symmetric matrix (recall that while a symmetric matrix is one such that $S = S^T$, an anti-symmetric matrix is one such that $A = -A^T$). *(10 points)*

(b) For a general matrix X, suppose you want to write it as $X = S + A$, where S is symmetric and A is anti-symmetric. Can you find formulas for S and A in terms of X only? *(5 points)*

Problem 5: (a) Consider the matrix:

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix}$$

and compute its LU factorization. *(10 points)*

(b) Based on the example in part (a), how do you think the LU factorization of a matrix of the form:

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 & b_6 & c_6 \\ 0 & 0 & 0 & 0 & 0 & a_7 & b_7 \end{bmatrix}$$

will look like? (Just a general guess on how the matrices L and U will look will suffice for now. Hint: L and U will have a lot of zeroes. Where do you think they are located?) *(5 points)*

(c) In the generality of part (b), work out recursive formulas for the coefficients of L and U in terms of $a_2, ..., a_7$, $b_1, ..., b_7$, $c_1, ..., c_6$. *(5 points)*